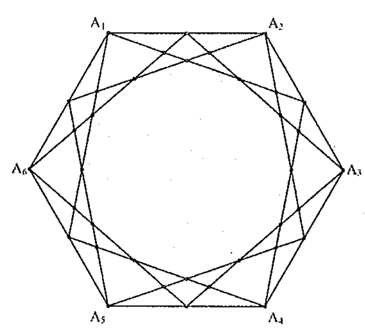
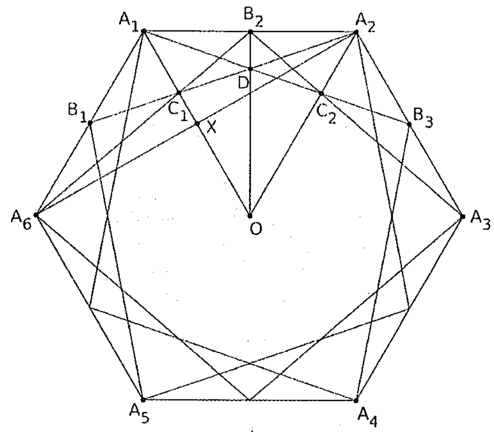
4537. *Proposed by Arsalan Wares.*

Let A be a regular hexagon with vertices A_1, A_2, A_3, A_4, A_5 and A_6. The six midpoints on the six sides of hexagon A are connected to the six vertices with 12 line segments as shown. The dodecagon formed by these 12 line segments has been shaded. What part of hexagon A has been shaded?



We received 21 submissions, out of which twelve were correct and complete. The incorrect submissions all overlooked that the dodecagon in the question is not regular. We present the solution by the Missouri State Problem Solving Group, lightly edited.



Let O be the center of the hexagon and B_1, B_2, B_3 be the midpoints of A_6A_1, A_1A_2 and A_2A_3. By symmetry, the lines A_6B_2, A_2B_1, and OA_1 intersect in a point, which we call C_1. Similarly we define C_2 as the intersection of A_1B_3, A_3B_2, and OA_2 and we let D be the intersection of A_2B_1, A_1B_3, and OB_2. Finally we define X as the intersection of A_6A_2 and OA_1. Since $OA_6A_1A_2$ is a rhombus, X is also the midpoint of OA_1.

The hexagon can be partitioned into twelve triangles that are all congruent to ΔOA_1B_2, whereas the dodecagon can be partitioned into twelve triangles, all congruent to ΔOC_1D. Therefore the ratio of the area of the dodecagon to that of the hexagon is equal to the ratio of the area of ΔOC_1D to that of ΔOA_1B_2.

Consider the triangle $A_1A_2A_6$. Since A_2B_1 and A_6B_2 are medians, C_1 is the centroid. Therefore $|A_1C_1| = \frac{2}{3}|XA_1| = \frac{1}{3}|OA_1|$. Similarly $|A_2C_2| = \frac{1}{3}|OA_2|$. Consider the triangle OA_1A_2. The point D is the intersection of the three cevians OB_2, A_1C_2, and A_2C_1. Therefore

$$\frac{OD}{DB_2} = \frac{OC_1}{C_1A_1} + \frac{OC_2}{C_2A_2} = 2 + 2 = 4,$$

and thus $|OD|/|OB_2| = 4/5$. Finally, using the sine law,

$$\begin{aligned}\frac{[OC_1D]}{[OA_1B_2]} &= \frac{\frac{1}{2}\cdot|OC_1|\cdot|OD|\cdot\sin\angle C_1OD}{\frac{1}{2}\cdot|OA_1|\cdot|OB_2|\cdot\sin\angle A_1OB_2}\\ &= \frac{|OC_1}{|OA_1|}\cdot\frac{|OD|}{|OB_2|}\\ &= \frac{2}{3}\cdot\frac{4}{5}\\ &= \frac{8}{15}.\end{aligned}$$

Therefore the ratio of the area of the dodecagon to the area of the hexagon is $8/15$.